Congressional Apportionment

Charles Biles, Ph.D.
Mathematics 103I: Contemporary Mathematics
Humboldt State University
February 2016

website: nia977.wix.com/drbcap

> "… no political problem is less susceptible of a precise solution than that which relates to the number most convenient for a representative legislature, …"
>
> James Madison
> *The Federalist 55*

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The Congressional Apportionment Problem

Determine how many seats in the U.S. House of Representatives each state gets.

CONGRESSIONAL SEATS

2010
OFFICIAL RESULTS

(US apportionment population = 309,183,463)/435 ≈ 710,767

http://www.census.gov/2010census/data/apportionment-data.php

The Constitution: Article I

Section 1. All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

The Constitution: Article I

Section 2. The House of Representatives shall be composed of Members **chosen every second Year by the People of the several States**, . . .

The Constitution: Article I

Section 2. The House of Representatives shall be composed of Members **chosen every second Year by the People of the several States**, . . .

Representatives . . . shall be apportioned among the several States . . . **according to their respective Numbers,** . . .

The Constitution: Article I

Section 2. The House of Representatives shall be composed of Members **chosen every second Year by the People of the several States**, . . .

Representatives . . . shall be apportioned among the several States . . . **according to their respective Numbers,** . . .

The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, **and within every subsequent Term of ten Years**, . . .

The Constitution: Article I

Section 2. The House of Representatives shall be composed of Members **chosen every second Year by the People of the several States**, . . .

Representatives . . . shall be apportioned among the several States . . . **according to their respective Numbers,** . . .

The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, **and within every subsequent Term of ten Years**, . . .

The Number of Representatives shall **not exceed one for every thirty Thousand,** but **each State shall have at Least one** Representative**;** . . .

First Apportionment Bills

Census 1790

State	Population
CT	236841
DE	55540
GA	70835
KY	68705
MD	278514
MA	475327
NH	141822
NJ	179570
NY	331589
NC	353523
PA	432879
RI	68446
SC	206236
VT	85533
VA	630560
US	3615920

First Apportionment Bills

Census 1790

State	Population
CT	236841
DE	55540
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MA	475327
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RI	68446
SC	206236
VT	85533
VA	630560
US	3615920

3792621 — City of Los Angeles 2010

First Apportionment Bills

Census 1790		House Bill
State	**Population**	**30000**
CT	236841	
DE	55540	
GA	70835	
KY	68705	
MD	278514	
MA	475327	
NH	141822	
NJ	179570	
NY	331589	
NC	353523	
PA	432879	
RI	68446	
SC	206236	
VT	85533	
VA	630560	
US	3615920	

First Apportionment Bills

Census 1790		House Bill
State	**Population**	**Divisor 30000**
CT	236841	
DE	55540	
GA	70835	
KY	68705	
MD	278514	
MA	475327	
NH	141822	
NJ	179570	
NY	331589	
NC	353523	
PA	432879	
RI	68446	
SC	206236	
VT	85533	
VA	630560	
US	3615920	

First Apportionment Bills

Census 1790		House Bill
State	**Population**	**Divisor 30000**
CT	236841	7.895
DE	55540	1.851
GA	70835	2.361
KY	68705	2.290
MD	278514	9.284
MA	475327	15.844
NH	141822	4.727
NJ	179570	5.986
NY	331589	11.053
NC	353523	11.784
PA	432879	14.429
RI	68446	2.282
SC	206236	6.875
VT	85533	2.851
VA	630560	21.019
US	3615920	

First Apportionment Bills

Census 1790		House Bill	
State	**Population**	**Divisor 30000**	**Seats**
CT	236841	7.895	7
DE	55540	1.851	1
GA	70835	2.361	2
KY	68705	2.290	2
MD	278514	9.284	9
MA	475327	15.844	15
NH	141822	4.727	4
NJ	179570	5.986	5
NY	331589	11.053	11
NC	353523	11.784	11
PA	432879	14.429	14
RI	68446	2.282	2
SC	206236	6.875	6
VT	85533	2.851	2
VA	630560	21.019	21
US	3615920		

First Apportionment Bills

Census 1790		House Bill	
State	**Population**	**Divisor 30000**	**Seats**
CT	236841	7.895	7
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NC	353523	11.784	11
PA	432879	14.429	14
RI	68446	2.282	2
SC	206236	6.875	6
VT	85533	2.851	2
VA	630560	21.019	21
US	3615920		**112**

First Apportionment Bills

Census 1790		House Bill		Senate Bill	
State	**Population**	**Divisor 30000**	**Seats**	**Divisor 33000**	**Seats**
CT	236841	7.895	7	7.177	7
DE	55540	1.851	1	1.683	1
GA	70835	2.361	2	2.147	2
KY	68705	2.290	2	2.082	2
MD	278514	9.284	9	8.440	8
MA	475327	15.844	15	14.404	14
NH	141822	4.727	4	4.298	4
NJ	179570	5.986	5	5.442	5
NY	331589	11.053	11	10.048	10
NC	353523	11.784	11	10.713	10
PA	432879	14.429	14	13.118	13
RI	68446	2.282	2	2.074	2
SC	206236	6.875	6	6.250	6
VT	85533	2.851	2	2.592	2
VA	630560	21.019	21	19.108	19
US	3615920		**112**		

First Apportionment Bills

Census 1790		House Bill		Senate Bill	
State	**Population**	**Divisor 30000**	**Seats**	**Divisor 33000**	**Seats**
CT	236841	7.895	7	7.177	7
DE	55540	1.851	1	1.683	1
GA	70835	2.361	2	2.147	2
KY	68705	2.290	2	2.082	2
MD	278514	9.284	9	8.440	8
MA	475327	15.844	15	14.404	14
NH	141822	4.727	4	4.298	4
NJ	179570	5.986	5	5.442	5
NY	331589	11.053	11	10.048	10
NC	353523	11.784	11	10.713	10
PA	432879	14.429	14	13.118	13
RI	68446	2.282	2	2.074	2
SC	206236	6.875	6	6.250	6
VT	85533	2.851	2	2.592	2
VA	630560	21.019	21	19.108	19
US	3615920		**112**		**105**

Rule of Three

Federalists in Congress apply a new idea:

Multiply the House size by each state's proportion to determine the state's **quota** (fair share of the House).

Rule of Three

Federalists in Congress apply a new idea:

Multiply the House size by each state's proportion to determine the state's **quota** (fair share of the House).

$$\textbf{\textit{quota}} = (House\ size) \times \frac{state\ population}{national\ population}$$

The House Bill

Census 1790		House Bill	
State	**Population**	**Divisor 30000**	**Seats**
CT	236841	7.895	7
DE	55540	1.851	1
GA	70835	2.361	2
KY	68705	2.290	2
MD	278514	9.284	9
MA	475327	15.844	15
NH	141822	4.727	4
NJ	179570	5.986	5
NY	331589	11.053	11
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RI	68446	2.282	2
SC	206236	6.875	6
VT	85533	2.851	2
VA	630560	21.019	21
US	3615920		

The House Bill

Census 1790		House Bill	
State	**Population**	**Divisor 30000**	**Seats**
CT	236841	7.895	7
DE	55540	1.851	1
GA	70835	2.361	2
KY	68705	2.290	2
MD	278514	9.284	9
MA	475327	15.844	15
NH	141822	4.727	4
NJ	179570	5.986	5
NY	331589	11.053	11
NC	353523	11.784	11
PA	432879	14.429	14
RI	68446	2.282	2
SC	206236	6.875	6
VT	85533	2.851	2
VA	630560	21.019	21
US	3615920		**112**

The House Bill

Census 1790		House Bill		
State	**Population**	**Divisor 30000**	**Seats**	***h*=112**
CT	236841	7.895	7	
DE	55540	1.851	1	
GA	70835	2.361	2	
KY	68705	2.290	2	
MD	278514	9.284	9	
MA	475327	15.844	15	
NH	141822	4.727	4	
NJ	179570	5.986	5	
NY	331589	11.053	11	
NC	353523	11.784	11	
PA	432879	14.429	14	
RI	68446	2.282	2	
SC	206236	6.875	6	
VT	85533	2.851	2	
VA	630560	21.019	21	
US	3615920		**112**	

The House Bill

Census 1790		House Bill			
State	Population	Divisor 30000	Seats	Quota	*h*=112
CT	236841	7.895	7		7.336
DE	55540	1.851	1		1.720
GA	70835	2.361	2		2.194
KY	68705	2.290	2		2.128
MD	278514	9.284	9		8.627
MA	475327	15.844	15		14.723
NH	141822	4.727	4		4.393
NJ	179570	5.986	5		5.562
NY	331589	11.053	11		10.271
NC	353523	11.784	11		10.950
PA	432879	14.429	14		13.408
RI	68446	2.282	2		2.120
SC	206236	6.875	6		6.388
VT	85533	2.851	2		2.649
VA	630560	21.019	21		19.531
US	3615920		**112**		112

Problem

Census 1790		House Bill		
State	Population	Divisor 30000	Seats	Quota *h*=112
CT	236841	7.895	7	7.336
DE	55540	1.851	1	1.720
GA	70835	2.361	2	2.194
KY	68705	2.290	2	2.128
MD	278514	9.284	9	8.627
MA	475327	15.844	15	14.723
NH	141822	4.727	4	4.393
NJ	179570	5.986	5	5.562
NY	331589	11.053	11	10.271
NC	353523	11.784	11	10.950
PA	432879	14.429	14	13.408
RI	68446	2.282	2	2.120
SC	206236	6.875	6	6.388
VT	85533	2.851	2	2.649
VA	630560	**21.019**	**21**	**19.531**
US	3615920		**112**	112

The Quota Rule is violated.

The Senate Bill

Census 1790		Senate Bill	
State	Population	Divisor 33000	Seats
CT	236841	7.177	7
DE	55540	1.683	1
GA	70835	2.147	2
KY	68705	2.082	2
MD	278514	8.440	8
MA	475327	14.404	14
NH	141822	4.298	4
NJ	179570	5.442	5
NY	331589	10.048	10
NC	353523	10.713	10
PA	432879	13.118	13
RI	68446	2.074	2
SC	206236	6.250	6
VT	85533	2.592	2
VA	630560	19.108	19
US	3615920		**105**

The Senate Bill

Census 1790		Senate Bill		
State	Population	Divisor 33000	Seats	Quota *h*=105
CT	236841	7.177	7	6.877
DE	55540	1.683	1	1.613
GA	70835	2.147	2	2.057
KY	68705	2.082	2	1.995
MD	278514	8.440	8	8.088
MA	475327	14.404	14	13.803
NH	141822	4.298	4	4.118
NJ	179570	5.442	5	5.214
NY	331589	10.048	10	9.629
NC	353523	10.713	10	10.266
PA	432879	13.118	13	12.570
RI	68446	2.074	2	1.988
SC	206236	6.250	6	5.989
VT	85533	2.592	2	2.484
VA	630560	19.108	19	18.310
US	3615920		105	105

Problem

	Census 1790	Senate Bill		
State	**Population**	**Divisor 33000**	**Seats**	**Quota *h*=105**
CT	236841	7.177	7	6.877
DE	**55540**	**1.683**	**1**	**1.613**
GA	70835	2.147	2	2.057
KY	68705	2.082	2	1.995
MD	278514	8.440	8	8.088
MA	475327	14.404	14	13.803
NH	141822	4.298	4	4.118
NJ	179570	5.442	5	5.214
NY	331589	10.048	10	9.629
NC	353523	10.713	10	10.266
PA	432879	13.118	13	12.570
RI	68446	2.074	2	1.988
SC	206236	6.250	6	5.989
VT	85533	2.592	2	2.484
VA	**630560**	**19.108**	**19**	**18.310**
US	3615920		105	105

Large states are favored over small states.

Hamilton's Method

State	Population	
CT	236841	
DE	55540	
GA	70835	
KY	68705	
MD	278514	
MA	475327	
NH	141822	
NJ	179570	
NY	331589	
NC	353523	
PA	432879	
RI	68446	
SC	206236	
VT	85533	
VA	630560	
US	3615920	**120.5307**

***d* = 30000**

Hamilton's Method

State	Population	
CT	236841	
DE	55540	
GA	70835	
KY	68705	
MD	278514	
MA	475327	
NH	141822	
NJ	179570	
NY	331589	
NC	353523	
PA	432879	
RI	68446	
SC	206236	
VT	85533	
VA	630560	
US	3615920	**120.5307**

***d* = 30000**

3615920/121 = **29883.6**

Hamilton's Method

State	Population	*h* = 120
CT	236841	
DE	55540	
GA	70835	
KY	68705	
MD	278514	
MA	475327	
NH	141822	
NJ	179570	
NY	331589	
NC	353523	
PA	432879	
RI	68446	
SC	206236	
VT	85533	
VA	630560	
US	3615920	**120.5307**

***d* = 30000**

Hamilton's Method

State	Population	*h* = 120	Quota
CT	236841		7.860
DE	55540		1.843
GA	70835		2.351
KY	68705		2.280
MD	278514		9.243
MA	475327		15.774
NH	141822		4.707
NJ	179570		5.959
NY	331589		11.004
NC	353523		11.732
PA	432879		14.366
RI	68446		2.271
SC	206236		6.844
VT	85533		2.839
VA	630560		20.926
US	3615920	**120.5307**	120

***d* = 30000**

Hamilton's Method

State	Population	*h* = 120	Quota	Lower Q
CT	236841		7.860	7
DE	55540		1.843	1
GA	70835		2.351	2
KY	68705		2.280	2
MD	278514		9.243	9
MA	475327		15.774	15
NH	141822		4.707	4
NJ	179570		5.959	5
NY	331589		11.004	11
NC	353523		11.732	11
PA	432879		14.366	14
RI	68446		2.271	2
SC	206236		6.844	6
VT	85533		2.839	2
VA	630560		20.926	20
US	3615920	**120.5307**	120	111

***d* = 30000**

Hamilton's Method

State	Population	*h* = 120	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
NJ	179570		5.959	5	6
NY	331589		11.004	11	11
NC	353523		11.732	11	12
PA	432879		14.366	14	14
RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	**120.5307**	120	111	120

***d* = 30000**

Hamilton's Method

State	Population	*h* = 120	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
NJ	179570		5.959	5	6
NY	331589		11.004	11	11
NC	353523		11.732	11	12
PA	432879		14.366	14	14
RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

This became the first apportionment bill passed by Congress.

Hamilton's Method

State	Population	*h* = 120	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
NJ	179570		5.959	5	6
NY	331589		11.004	11	11
NC	353523		11.732	11	12
PA	432879		14.366	14	14
RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

This became the first apportionment bill passed by Congress.

26 March 1792:
bill is sent to President Washington for his approval.

Hamilton's Method

State	Population	*h* = 120	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
NJ	179570		5.959	5	6
NY	331589		11.004	11	11
NC	353523		11.732	11	12
PA	432879		14.366	14	14
RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

This became the first apportionment bill passed by Congress.

26 March 1792:
bill is sent to President Washington for his approval.

5 April 1792: Washington vetoes the bill.

Hamilton's Method

State	Population	*h* = 120	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
NJ	179570		5.959	5	6
NY	331589		11.004	11	11
NC	353523		11.732	11	12
PA	432879		14.366	14	14
RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

U.S.:
3615920/120 = 30,132.66…

Hamilton's Method

State	Population	*h* = 120	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
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RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

Connecticut:
236841/8 = 29605.13 .

Delaware:
55540/2 = 27770

U.S.:
3615920/120 = 30,132.66…

Basic Jefferson Method

After Washington's veto on 5 April 1792, Congress quickly passed the original Senate bill. Washington signed the bill on 14 April 1972.

Two Methodologies

- Divisor Methods

- Quota Methods

Two Methodologies

- Divisor Methods
  - Basic
  - Modified
- Quota Methods

Two Methodologies

- Divisor Methods
  - Basic: *h* is the result
  - Modified
- Quota Methods

Two Methodologies

- Divisor Methods
  - Basic: *h* is the result
  - Modified: *h* is the goal
- Quota Methods

Two Methodologies

- Divisor Methods
  - Basic: *h* is the result
  - Modified: *h* is the goal
- Quota Methods

  h is the resource

Two Methodologies

- Divisor Methods
  - Basic: *h* is the result
  - Modified: *h* is the goal
- Quota Methods

  h is the resource

Divisor methods **create** seats.
Quota methods **distribute** seats.

Basic Jefferson Method

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The resulting house size is the sum of each state's apportionment.

First 60 years

- A Basic Divisor Method would be used as the House apportionment method until 1850.
  - 1790: $s = 15$; $d = 33000 \Rightarrow h = 105$
  - 1800: $s = 16$; $d = 33000 \Rightarrow h = 141$
  - 1810: $s = 17$; $d = 35000 \Rightarrow h = 181$
  - 1820: $s = 24$; $d = 40000 \Rightarrow h = 213$
  - 1830: $s = 24$; $d = 47700 \Rightarrow h = 240$
  - 1840: $s = 26$; $d = 70680 \Rightarrow h = 223$

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Webster: round normally.

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With 6 seats the constituency is 280,657/6 = 46,776.

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With 6 seats the constituency is 280,657/6 = 46,776.

A constituency of 46,776 is closer to the target constituency of 50,000; hence, Dean awards Vermont 6 seats.

James Dean

Step 1: Select the constituency, d.

Step 2: Calculate $q = p/d$ and $n = \text{int}(q)$.

Step 3: Let the apportionment be either n or $n+1$, with $n+1$ iff $p/(n+1)$ is closer to d than p/n.

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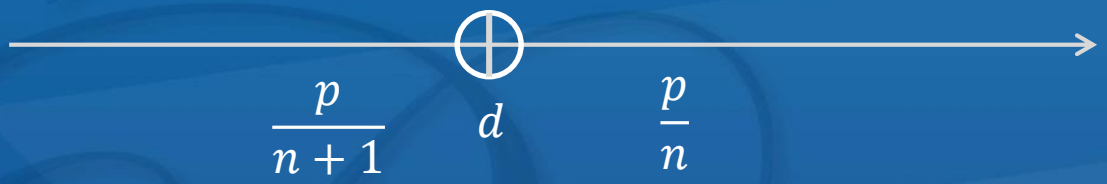

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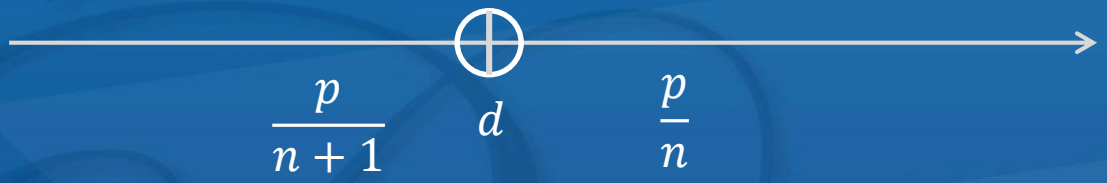

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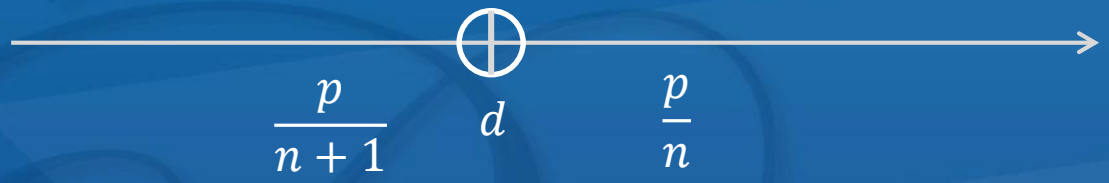

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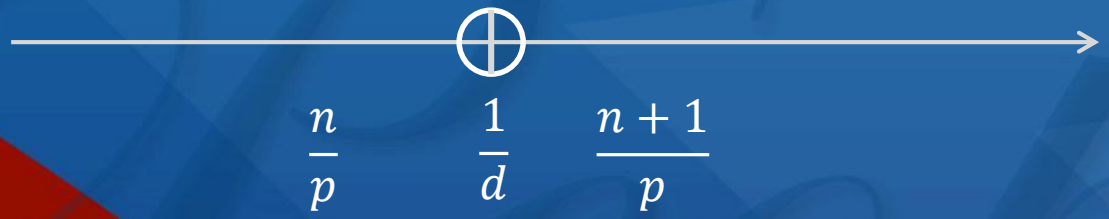

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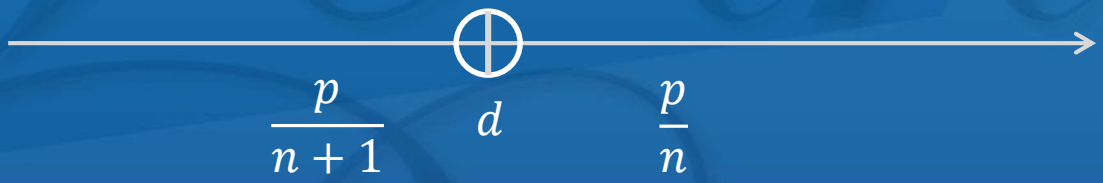

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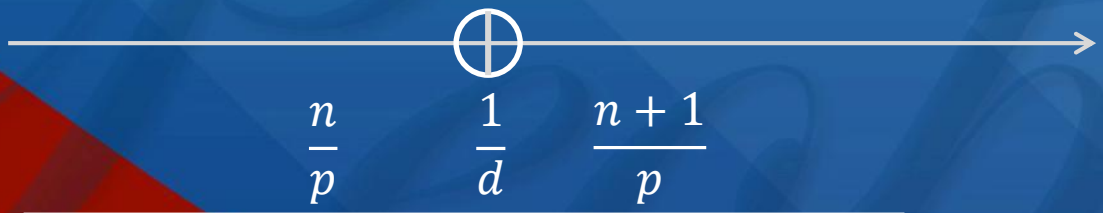

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In a Round About Way

Census 1810		d = 35000				
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GA	210346	6.0099	6	6	6	7
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261818/7 = 37403
Difference = 2403

261818/8 = 32727
Difference = 2273

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1840 Census

The Apportionment Act of 1842 used a basic divisor method with $d = 70680$ and **Webster's method** of rounding, yielding $h = 233$.

This was the only time in U.S. history that the House size decreased as a result of a census-based re-apportionment.

The Vinton Act

The Vinton Act of 1850 (Representative Samuel Vinton, Whig-Ohio) was passed to head off politicizing the census figures. The idea was to adopt a permanent appropriation act.

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But experience exposed problems with the Vinton Act.

Alabama Paradox

This Paradox may occur with the Hamilton method:

when the number of House seats is increased, a given state's apportion may decrease.

1910

Apportionment based on the 1910 census came from another mutation in apportionment methodology.

Congress abandoned the Quota Method and used a modified divisor method.

Modified Divisor Methods

Step 1. Select the House size, h.

Step 2. Select a constituency, d.

Step 3. Calculate $q = p/d$.

Step 4. Round the state's quotient to get a.

Step 5. If the apportionments add up to h,
then DONE;
else, modify d and GO TO Step 3.

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1910 result: $h = 433$ and Webster's method.

1920 Census

In the 1920 decade, for the only time in U. S. History, no census-based re-apportionment act was passed.

Congress could not agree on either the size of the House or on the method of apportionment. Further, the politics of prohibition played a significant role: the dries would not support any proposal that gave the wets more power.

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The Huntington-Hill method used today is a divisor method:

Let $q = p/d$ and $n = \text{int}(q)$.

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Webster: arithmetic mean $\frac{n+(n+1)}{2} = n + \frac{1}{2}$

Huntington-Hill: geometric mean $\sqrt{n(n+1)}$

Dean: harmonic mean $\frac{2}{\frac{1}{n}+\frac{1}{n+1}} = \frac{2n(n+1)}{2n+1}$

The Aftermath

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There are no perfect apportionment methods.

Any method that satisfies the quota rule produces paradoxes; any method that is free of the Alabama paradox may violate the quota rule.

Thank You

It is time that I took my seat in this House!

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