

Congressional Apportionment

Charles Biles, Ph.D.
University of Alaska Southeast
September 2017

website: nia977.wix.com/drbcap

“... no political problem is less susceptible of a precise solution than that which relates to the number most convenient for a representative legislature, ...”

James Madison
The Federalist 55

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The Congressional Apportionment Problem

Determine how many seats in the U.S.
House of Representatives each state gets.

Constitutional Guidelines

- Representatives shall be apportioned among the States according to their respective numbers as enumerated by a decennial census.
- Each State shall have at least one representative.
- The number of representatives shall not exceed one for every thirty thousand.

Constitutional Guidelines

The first enumeration shall be made within three years of the first meeting of Congress.

... And until such enumeration shall be made, the State of New Hampshire shall be entitled to chuse three, Massachusetts eight, Rhode-Island and Providence Plantations one, Connecticut five, New-York six, New Jersey four, Pennsylvania eight, Delaware one, Maryland six, Virginia ten, North Carolina five, South Carolina five, and Georgia three.

CONGRESSIONAL SEATS

2010
OFFICIAL RESULTS



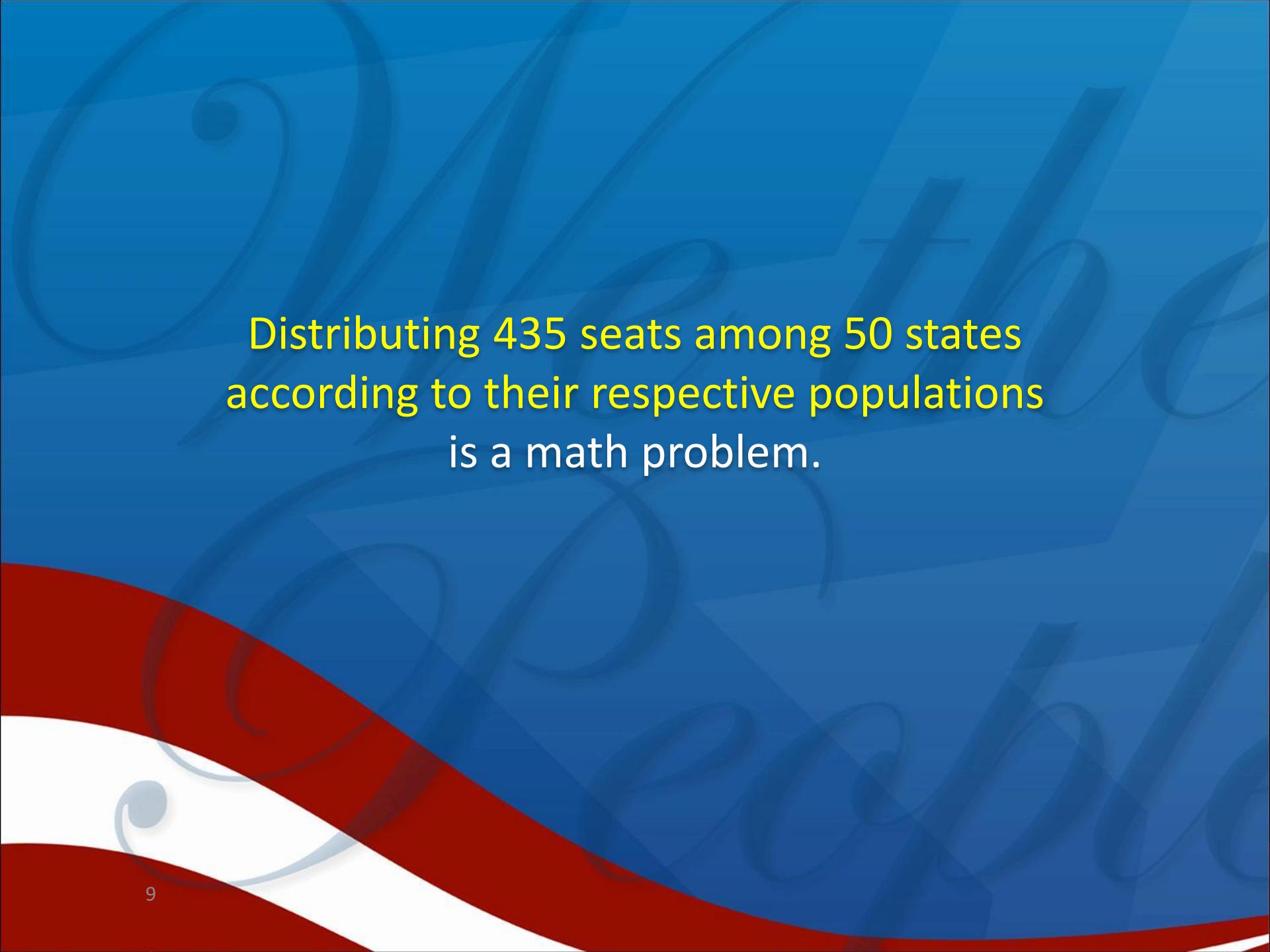
GAINED
LOST
NO CHANGE

United States
Census
2010

(US apportionment population = 309,183,463)/435 ≈ 710,767

<http://www.census.gov/2010census/data/apportionment-data.php>

Note: $50 \leq h \leq 10306$ by constitutional constraints.



Distributing 435 seats among 50 states
according to their respective populations
is a math problem.

Two Math Skills

1. Average two different positive numbers.
2. Round a positive decimal number.

Two Math Skills

1. The average of a and b where $0 < a < b$.

Two Math Skills

1. The average of a and b where $0 < a < b$.

$$\text{ave}(a,b) = \max(a,b) = b$$

$$\min(a,b) = a$$

$$\text{AM}(a,b) = (a + b)/2$$

Two Math Skills

1. The average of a and b where $0 < a < b$.

$$\text{ave}(a,b) = \max(a,b) = b$$

$$\min(a,b) = a$$

$$\text{AM}(a,b) = (a + b)/2$$

$$\text{HM}(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a + b}$$

Two Math Skills

1. The average of a and b where $0 < a < b$.

$$\text{ave}(a,b) = \max(a,b) = b$$

$$\min(a,b) = a$$

$$\text{AM}(a,b) = (a + b)/2$$

$$\text{HM}(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a + b}$$

$$\text{GM}(a,b) = \sqrt{a \times b}$$

Two Math Skills

For Example: $\text{ave}(8,12) =$

- $\max(8,12) = 12$
- $\min(8,12) = 8$
- $\text{AM}(8,12) = 10$
- $\text{HM}(8,12) = \frac{2}{\frac{1}{8} + \frac{1}{12}} = 9.6$
- $\text{GM}(8,12) = \sqrt{8 \times 12} \approx 9.8$

Two Math Skills

For Example: $\text{ave}(8,12) =$

- $\max(8,12) = 12$
- $\min(8,12) = 8$
- $\text{AM}(8,12) = 10$
- $\text{HM}(8,12) = \frac{2}{\frac{1}{8} + \frac{1}{12}} = 9.6$
- $\text{GM}(8,12) = \sqrt{8 \times 12} \approx 9.8$

Always, $\min < \text{HM} < \text{GM} < \text{AM} < \max$.

<http://nia977.wixsite.com/drbcap/resources>

An Average Lesson

Two Math Skills

2. How to round a positive decimal number.

Two Math Skills

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Suppose q is a positive decimal number, not an integer, whose integer part is n so $n < q < n+1$.

Two Math Skills

2. How to round a positive decimal number.

Suppose q is a positive decimal number, not an integer,
whose integer part is n so $n < q < n+1$.

Then $\text{round}(q)$ is either n or $n+1$ where $\text{round}(q) = n+1$
if and only if

Two Math Skills

2. How to round a positive decimal number.

Suppose q is a positive decimal number, not an integer,
whose integer part is n so $n < q < n+1$.

Then $\text{round}(q)$ is either n or $n+1$ where $\text{round}(q) = n+1$
if and only if $q \geq \text{ave}(n,n+1)$:

Two Math Skills

2. How to round a positive decimal number.

Suppose q is a positive decimal number, not an integer, whose integer part is n so $n < q < n+1$.

Then $\text{round}(q)$ is either n or $n+1$ where $\text{round}(q) = n+1$ if and only if $q \geq \text{ave}(n,n+1)$:

| | |
|------------------------|----------------|
| $q \geq \max(n,n+1)$ | round down |
| $q < \min(n,n+1)$ | round up |
| $q = \text{AM}(n,n+1)$ | round normally |
| $q = \text{GM}(n,n+1)$ | geometric mean |
| $q = \text{HM}(n,n+1)$ | harmonic mean |

Congressional Apportionment

Two Approaches

- Constituency Approach:
- House Size Approach:

Congressional Apportionment

Two Approaches

- **Constituency Approach:** How many people should a congressperson represent?
- **House Size Approach:**

Congressional Apportionment

Two Approaches

- **Constituency Approach:** How many people should a congressperson represent?
- **House Size Approach:** How many seats should there be in the House?

First Apportionment Bills

Census 1790

| State | Population |
|-------|------------|
| CT | 236841 |
| DE | 55540 |
| GA | 70835 |
| KY | 68705 |
| MD | 278514 |
| MA | 475327 |
| NH | 141822 |
| NJ | 179570 |
| NY | 331589 |
| NC | 353523 |
| PA | 432879 |
| RI | 68446 |
| SC | 206236 |
| VT | 85533 |
| VA | 630560 |
| US | 3615920 |

First Apportionment Bills

Census 1790

| State | Population |
|-------|------------|
| CT | 236841 |
| DE | 55540 |
| GA | 70835 |
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3792621 — City of Los Angeles 2010

First Apportionment Bills

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House Bill

30000

First Apportionment Bills

Census 1790

| State | Population |
|-------|------------|
| CT | 236841 |
| DE | 55540 |
| GA | 70835 |
| KY | 68705 |
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House Bill

Divisor 30000

First Apportionment Bills

| Census 1790 | | House Bill |
|-------------|------------|---------------|
| State | Population | Divisor 30000 |
| CT | 236841 | 7.895 |
| DE | 55540 | 1.851 |
| GA | 70835 | 2.361 |
| KY | 68705 | 2.290 |
| MD | 278514 | 9.284 |
| MA | 475327 | 15.844 |
| NH | 141822 | 4.727 |
| NJ | 179570 | 5.986 |
| NY | 331589 | 11.053 |
| NC | 353523 | 11.784 |
| PA | 432879 | 14.429 |
| RI | 68446 | 2.282 |
| SC | 206236 | 6.875 |
| VT | 85533 | 2.851 |
| VA | 630560 | 21.019 |
| US | 3615920 | |

First Apportionment Bills

Census 1790

| State | Population |
|-------|------------|
| CT | 236841 |
| DE | 55540 |
| GA | 70835 |
| KY | 68705 |
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| PA | 432879 |
| RI | 68446 |
| SC | 206236 |
| VT | 85533 |
| VA | 630560 |
| US | 3615920 |

House Bill

| Divisor | 30000 | Seats |
|---------|---------|-------|
| 7.895 | | 7 |
| 1.851 | | 1 |
| 2.361 | | 2 |
| 2.290 | | 2 |
| 9.284 | | 9 |
| 15.844 | | 15 |
| 4.727 | | 4 |
| 5.986 | | 5 |
| 11.053 | | 11 |
| 11.784 | | 11 |
| 14.429 | | 14 |
| 2.282 | | 2 |
| 6.875 | | 6 |
| 2.851 | | 2 |
| 21.019 | | 21 |
| US | 3615920 | |

First Apportionment Bills

| Census 1790 | | House Bill | | |
|-------------|------------|------------|-------|-------|
| State | Population | Divisor | 30000 | Seats |
| CT | 236841 | 7.895 | | 7 |
| DE | 55540 | 1.851 | | 1 |
| GA | 70835 | 2.361 | | 2 |
| KY | 68705 | 2.290 | | 2 |
| MD | 278514 | 9.284 | | 9 |
| MA | 475327 | 15.844 | | 15 |
| NH | 141822 | 4.727 | | 4 |
| NJ | 179570 | 5.986 | | 5 |
| NY | 331589 | 11.053 | | 11 |
| NC | 353523 | 11.784 | | 11 |
| PA | 432879 | 14.429 | | 14 |
| RI | 68446 | 2.282 | | 2 |
| SC | 206236 | 6.875 | | 6 |
| VT | 85533 | 2.851 | | 2 |
| VA | 630560 | 21.019 | | 21 |
| US | 3615920 | 112 | | |

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| Divisor 30000 | Seats |
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| 1.851 | 1 |
| 2.361 | 2 |
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| 9.284 | 9 |
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| 11.784 | 11 |
| 14.429 | 14 |
| 2.282 | 2 |
| 6.875 | 6 |
| 2.851 | 2 |
| 21.019 | 21 |
| 112 | |

Senate Bill

| Divisor 33000 | Seats |
|---------------|-------|
| 7.177 | 7 |
| 1.683 | 1 |
| 2.147 | 2 |
| 2.082 | 2 |
| 8.440 | 8 |
| 14.404 | 14 |
| 4.298 | 4 |
| 5.442 | 5 |
| 10.048 | 10 |
| 10.713 | 10 |
| 13.118 | 13 |
| 2.074 | 2 |
| 6.250 | 6 |
| 2.592 | 2 |
| 19.108 | 19 |

First Apportionment Bills

Census 1790

| State | Population |
|-------|------------|
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| 10.713 | 10 |
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| 2.592 | 2 |
| 19.108 | 19 |
| 105 | |

Rule of Three

Federalists in Congress apply a new idea:

Multiply the House size by each state's proportion to determine the state's **quota** (fair share of the House).

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$$\text{quota} = (\text{House size}) \times \frac{\text{state population}}{\text{national population}}$$

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Multiply the House size by each state's proportion to determine the state's **quota** (fair share of the House).

$$\text{quota} = (\text{House size}) \times \frac{\text{state population}}{\text{national population}}$$

Rule of Three

The House Bill

Census 1790

House Bill

| State | Population | Divisor 30000 | Seats |
|-------|------------|---------------|-------|
| CT | 236841 | 7.895 | 7 |
| DE | 55540 | 1.851 | 1 |
| GA | 70835 | 2.361 | 2 |
| KY | 68705 | 2.290 | 2 |
| MD | 278514 | 9.284 | 9 |
| MA | 475327 | 15.844 | 15 |
| NH | 141822 | 4.727 | 4 |
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| NY | 331589 | 11.053 | 11 |
| NC | 353523 | 11.784 | 11 |
| PA | 432879 | 14.429 | 14 |
| RI | 68446 | 2.282 | 2 |
| SC | 206236 | 6.875 | 6 |
| VT | 85533 | 2.851 | 2 |
| VA | 630560 | 21.019 | 21 |
| US | 3615920 | | |

The House Bill

Census 1790

House Bill

| State | Population | Divisor 30000 | Seats |
|-------|------------|---------------|-------|
| CT | 236841 | 7.895 | 7 |
| DE | 55540 | 1.851 | 1 |
| GA | 70835 | 2.361 | 2 |
| KY | 68705 | 2.290 | 2 |
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| SC | 206236 | 6.875 | 6 |
| VT | 85533 | 2.851 | 2 |
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| VT | 85533 |
| VA | 630560 |
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House Bill

| Divisor 30000 | Seats |
|---------------|-------|
| 7.895 | 7 |
| 1.851 | 1 |
| 2.361 | 2 |
| 2.290 | 2 |
| 9.284 | 9 |
| 15.844 | 15 |
| 4.727 | 4 |
| 5.986 | 5 |
| 11.053 | 11 |
| 11.784 | 11 |
| 14.429 | 14 |
| 2.282 | 2 |
| 6.875 | 6 |
| 2.851 | 2 |
| 21.019 | 21 |
| 112 | |

$h=112$

The House Bill

Census 1790

House Bill

| State | Population | Divisor 30000 | Seats | Quota $h=112$ |
|-------|------------|---------------|-------|---------------|
| CT | 236841 | 7.895 | 7 | 7.336 |
| DE | 55540 | 1.851 | 1 | 1.720 |
| GA | 70835 | 2.361 | 2 | 2.194 |
| KY | 68705 | 2.290 | 2 | 2.128 |
| MD | 278514 | 9.284 | 9 | 8.627 |
| MA | 475327 | 15.844 | 15 | 14.723 |
| NH | 141822 | 4.727 | 4 | 4.393 |
| NJ | 179570 | 5.986 | 5 | 5.562 |
| NY | 331589 | 11.053 | 11 | 10.271 |
| NC | 353523 | 11.784 | 11 | 10.950 |
| PA | 432879 | 14.429 | 14 | 13.408 |
| RI | 68446 | 2.282 | 2 | 2.120 |
| SC | 206236 | 6.875 | 6 | 6.388 |
| VT | 85533 | 2.851 | 2 | 2.649 |
| VA | 630560 | 21.019 | 21 | 19.531 |
| US | 3615920 | 112 | | 112 |

Problem

Census 1790

House Bill

| State | Population | Divisor 30000 | Seats | Quota $h=112$ |
|-------|------------|---------------|-------|---------------|
| CT | 236841 | 7.895 | 7 | 7.336 |
| DE | 55540 | 1.851 | 1 | 1.720 |
| GA | 70835 | 2.361 | 2 | 2.194 |
| KY | 68705 | 2.290 | 2 | 2.128 |
| MD | 278514 | 9.284 | 9 | 8.627 |
| MA | 475327 | 15.844 | 15 | 14.723 |
| NH | 141822 | 4.727 | 4 | 4.393 |
| NJ | 179570 | 5.986 | 5 | 5.562 |
| NY | 331589 | 11.053 | 11 | 10.271 |
| NC | 353523 | 11.784 | 11 | 10.950 |
| PA | 432879 | 14.429 | 14 | 13.408 |
| RI | 68446 | 2.282 | 2 | 2.120 |
| SC | 206236 | 6.875 | 6 | 6.388 |
| VT | 85533 | 2.851 | 2 | 2.649 |
| VA | 630560 | 21.019 | 21 | 19.531 |
| US | 3615920 | 112 | | 112 |

The Quota Rule
is violated.

The Senate Bill

Census 1790

| State | Population |
|-------|------------|
| CT | 236841 |
| DE | 55540 |
| GA | 70835 |
| KY | 68705 |
| MD | 278514 |
| MA | 475327 |
| NH | 141822 |
| NJ | 179570 |
| NY | 331589 |
| NC | 353523 |
| PA | 432879 |
| RI | 68446 |
| SC | 206236 |
| VT | 85533 |
| VA | 630560 |
| US | 3615920 |

Senate Bill

| Divisor 33000 | Seats | Quota $h=105$ |
|---------------|-------|---------------|
| 7.177 | 7 | 6.877 |
| 1.683 | 1 | 1.613 |
| 2.147 | 2 | 2.057 |
| 2.082 | 2 | 1.995 |
| 8.440 | 8 | 8.088 |
| 14.404 | 14 | 13.803 |
| 4.298 | 4 | 4.118 |
| 5.442 | 5 | 5.214 |
| 10.048 | 10 | 9.629 |
| 10.713 | 10 | 10.266 |
| 13.118 | 13 | 12.570 |
| 2.074 | 2 | 1.988 |
| 6.250 | 6 | 5.989 |
| 2.592 | 2 | 2.484 |
| 19.108 | 19 | 18.310 |
| | 105 | 105 |



No
Quota Rule
Violation

Problem

Census 1790

Senate Bill

| State | Population | Divisor 33000 | Seats | Quota $h=105$ |
|-------|---------------|---------------|-----------|---------------|
| CT | 236841 | 7.177 | 7 | 6.877 |
| DE | 55540 | 1.683 | 1 | 1.613 |
| GA | 70835 | 2.147 | 2 | 2.057 |
| KY | 68705 | 2.082 | 2 | 1.995 |
| MD | 278514 | 8.440 | 8 | 8.088 |
| MA | 475327 | 14.404 | 14 | 13.803 |
| NH | 141822 | 4.298 | 4 | 4.118 |
| NJ | 179570 | 5.442 | 5 | 5.214 |
| NY | 331589 | 10.048 | 10 | 9.629 |
| NC | 353523 | 10.713 | 10 | 10.266 |
| PA | 432879 | 13.118 | 13 | 12.570 |
| RI | 68446 | 2.074 | 2 | 1.988 |
| SC | 206236 | 6.250 | 6 | 5.989 |
| VT | 85533 | 2.592 | 2 | 2.484 |
| VA | 630560 | 19.108 | 19 | 18.310 |
| US | 3615920 | | 105 | 105 |

Large states are favored over small states.



Hamilton's Method

| State | Population |
|-------|------------|
| CT | 236841 |
| DE | 55540 |
| GA | 70835 |
| KY | 68705 |
| MD | 278514 |
| MA | 475327 |
| NH | 141822 |
| NJ | 179570 |
| NY | 331589 |
| NC | 353523 |
| PA | 432879 |
| RI | 68446 |
| SC | 206236 |
| VT | 85533 |
| VA | 630560 |
| US | 3615920 |

120.5307

$d = 30000$

Hamilton's Method

| State | Population | |
|-------|------------|--|
| CT | 236841 | |
| DE | 55540 | |
| GA | 70835 | |
| KY | 68705 | |
| MD | 278514 | |
| MA | 475327 | |
| NH | 141822 | |
| NJ | 179570 | |
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| NC | 353523 | |
| PA | 432879 | |
| RI | 68446 | |
| SC | 206236 | |
| VT | 85533 | |
| VA | 630560 | |
| US | 3615920 | |

$h = 120$

120.5307

$d = 30000$

Hamilton's Method

| State | Population | $h = 120$ | Quota |
|-------|------------|-----------------|--------|
| CT | 236841 | | 7.860 |
| DE | 55540 | | 1.843 |
| GA | 70835 | | 2.351 |
| KY | 68705 | | 2.280 |
| MD | 278514 | | 9.243 |
| MA | 475327 | | 15.774 |
| NH | 141822 | | 4.707 |
| NJ | 179570 | | 5.959 |
| NY | 331589 | | 11.004 |
| NC | 353523 | | 11.732 |
| PA | 432879 | | 14.366 |
| RI | 68446 | | 2.271 |
| SC | 206236 | | 6.844 |
| VT | 85533 | | 2.839 |
| VA | 630560 | | 20.926 |
| US | 3615920 | 120.5307 | 120 |

$$= 120 \times \frac{236841}{3615920}$$

$d = 30000$

Hamilton's Method

| State | Population | $h = 120$ | Quota | Lower Q |
|-------|------------|-----------------|--------|---------|
| CT | 236841 | | 7.860 | 7 |
| DE | 55540 | | 1.843 | 1 |
| GA | 70835 | | 2.351 | 2 |
| KY | 68705 | | 2.280 | 2 |
| MD | 278514 | | 9.243 | 9 |
| MA | 475327 | | 15.774 | 15 |
| NH | 141822 | | 4.707 | 4 |
| NJ | 179570 | | 5.959 | 5 |
| NY | 331589 | | 11.004 | 11 |
| NC | 353523 | | 11.732 | 11 |
| PA | 432879 | | 14.366 | 14 |
| RI | 68446 | | 2.271 | 2 |
| SC | 206236 | | 6.844 | 6 |
| VT | 85533 | | 2.839 | 2 |
| VA | 630560 | | 20.926 | 20 |
| US | 3615920 | 120.5307 | 120 | 111 |

$d = 30000$

Hamilton's Method

| State | Population | $h = 120$ | Quota | Lower Q | Appt |
|-------|------------|-----------|--------|---------|------|
| CT | 236841 | | 7.860 | 7 | 8 |
| DE | 55540 | | 1.843 | 1 | 2 |
| GA | 70835 | | 2.351 | 2 | 2 |
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| MD | 278514 | | 9.243 | 9 | 9 |
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| VT | 85533 | | 2.839 | 2 | 3 |
| VA | 630560 | | 20.926 | 20 | 21 |
| US | 3615920 | 120.5307 | 120 | 111 | 120 |

$d = 30000$

Hamilton's Method

| State | Population | $h = 120$ | Quota | Lower Q | Appt |
|-------|------------|-----------|--------|---------|------|
| CT | 236841 | | 7.860 | 7 | 8 |
| DE | 55540 | | 1.843 | 1 | 2 |
| GA | 70835 | | 2.351 | 2 | 2 |
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5 April 1792: Washington vetoes the bill.

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U.S.:

$$3615920/120 = 30,132.66\dots$$

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Connecticut:
 $236841/8 = 29605.13$.

Delaware:
 $55540/2 = 27770$

U.S.:
 $3615920/120 = 30,132.66...$

First Apportionment Act

After Washington's veto on 5 April 1792, Congress quickly passed the original Senate bill. Washington signed the bill on 14 April 1792.

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The resulting House size is the sum of each state's apportionment.

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Webster: round normally.

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A constituency of 46,776 is closer to the target constituency of 50,000; hence, Dean awards Vermont 6 seats.

James Dean

Step 1: Select the constituency, d .

Step 2: Calculate $q = p/d$ and $n = \text{int}(q)$.

Step 3: Let the apportionment be either n or $n+1$,
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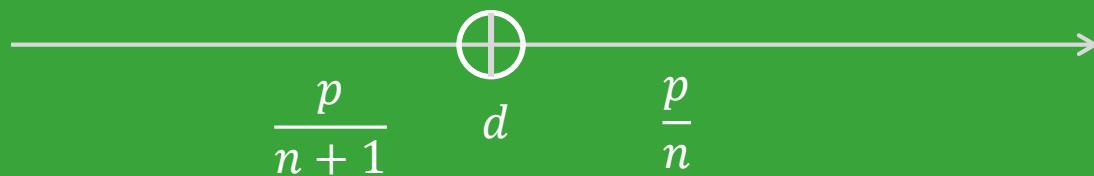
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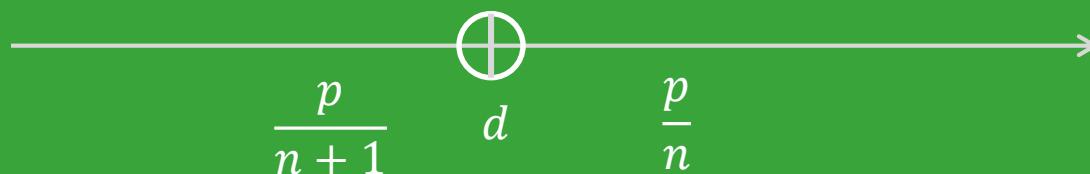


This is mathematically equivalent to: let the apportionment be $n + 1$ iff $q \geq \text{HM}(n, n + 1)$.

Dean's Criterion

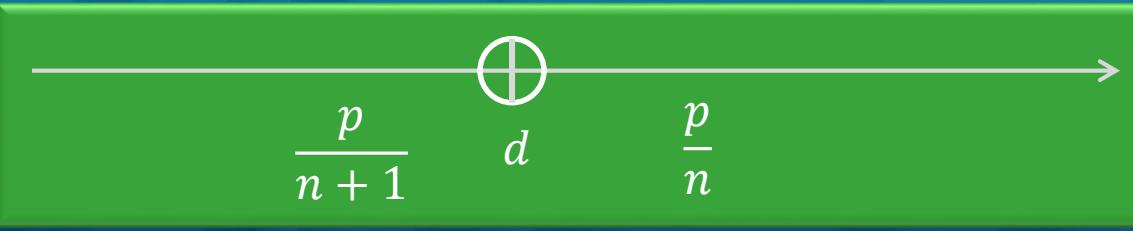
$$\frac{p}{n+1} \quad d \quad \frac{p}{n}$$


Dean's Criterion

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$$a = n + 1 \iff d - \frac{p}{n+1} \leq \frac{p}{n} - d$$

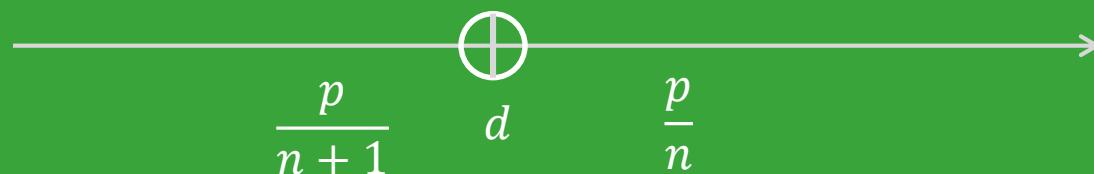
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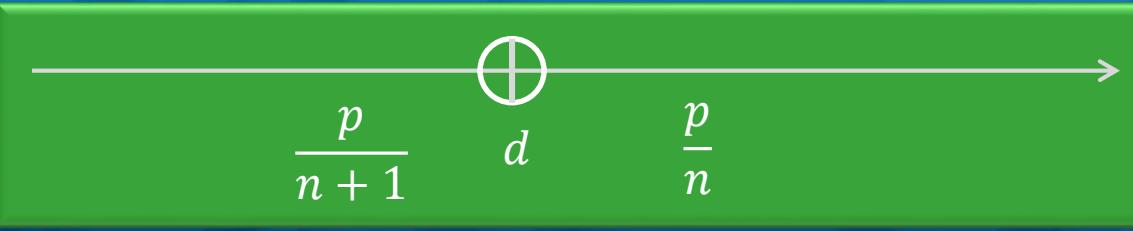
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Dean's Criterion



$$\begin{aligned} a = n + 1 &\iff d - \frac{p}{n+1} \leq \frac{p}{n} - d \\ &\iff 2d \leq \frac{p}{n} + \frac{p}{n+1} \\ &\iff 2d \leq p \left(\frac{1}{n} + \frac{1}{n+1} \right) \end{aligned}$$

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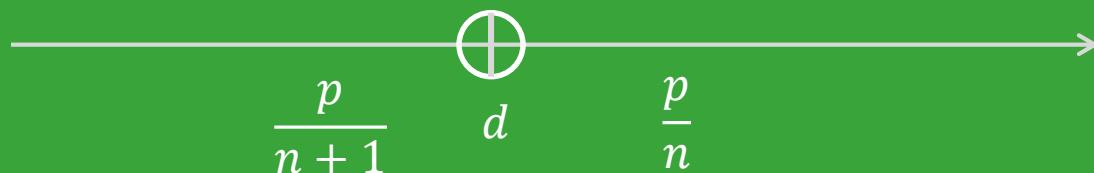
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$$\iff \frac{2}{\frac{1}{n} + \frac{1}{n+1}} \leq \frac{p}{d}$$

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Daniel Webster

Step 1: Select the constituency, d .

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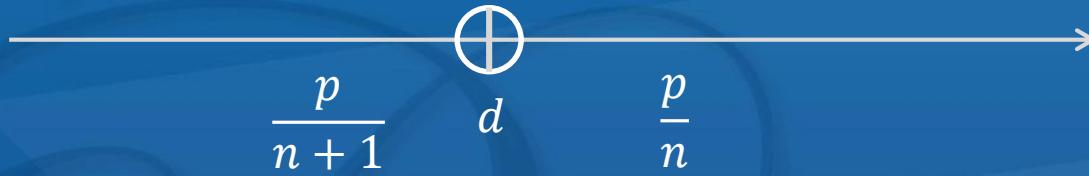
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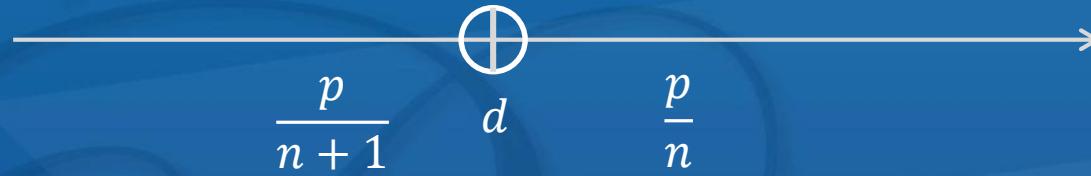
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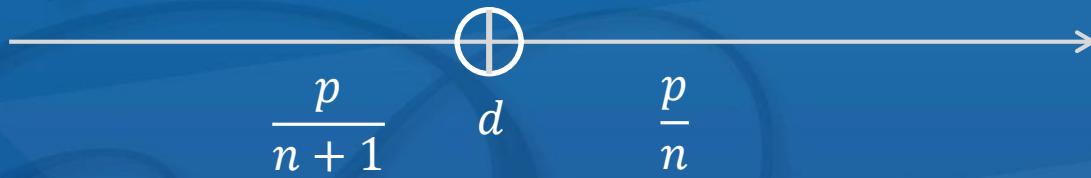
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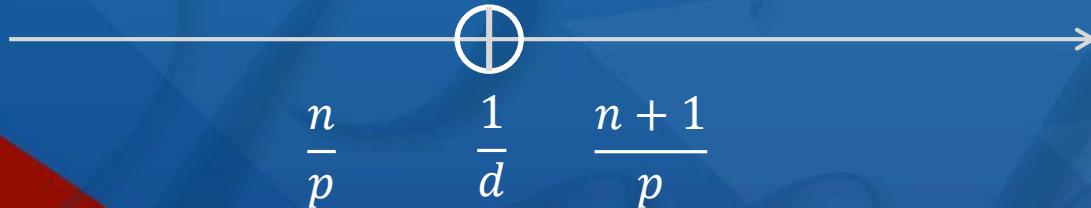
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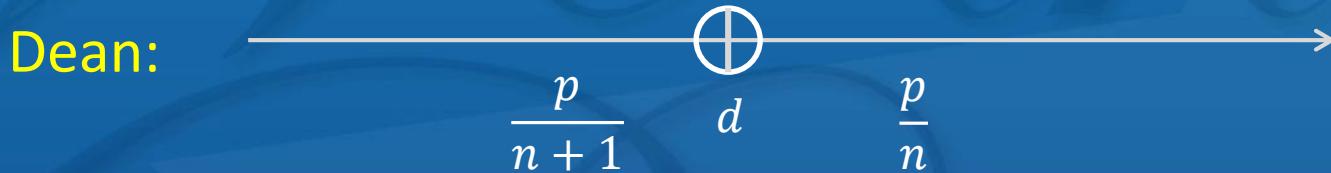


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In a Round About Way

| Census 1810 | | $d = 35000$ | | | | |
|-------------|------------|-------------|-----|-----|-----|-----|
| State | Population | Quotient | min | AM | HM | max |
| CT | 261818 | 7.4805 | 7 | 7 | 8 | 8 |
| DE | 71004 | 2.0287 | 2 | 2 | 2 | 3 |
| GA | 210346 | 6.0099 | 6 | 6 | 6 | 7 |
| KY | 374287 | 10.6939 | 10 | 11 | 11 | 11 |
| MD | 335946 | 9.5985 | 9 | 10 | 10 | 10 |
| MA | 700745 | 20.0213 | 20 | 20 | 20 | 21 |
| NH | 214460 | 6.1274 | 6 | 6 | 6 | 7 |
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| PA | 809773 | 23.1364 | 23 | 23 | 23 | 24 |
| RI | 76931 | 2.1980 | 2 | 2 | 2 | 3 |
| SC | 336569 | 9.6163 | 9 | 10 | 10 | 10 |
| TN | 243913 | 6.9689 | 6 | 7 | 7 | 7 |
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| SC | 336569 | 9.6163 | 9 | 10 | 10 | 10 |
| TN | 243913 | 6.9689 | 6 | 7 | 7 | 7 |
| VT | 217895 | 6.2256 | 6 | 6 | 6 | 7 |
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| US | 6575234 | 188.1222 | 181 | 188 | 189 | 198 |

In a Round About Way

| Census 1810 | | $d = 35000$ | | | | |
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| CT | 261818 | 7.4805 | 7 | 7 | 8 | 8 |
| DE | 71004 | 2.0287 | 2 | 2 | 2 | 3 |
| GA | 210346 | 6.0099 | 6 | 6 | 6 | 7 |
| KY | 374287 | 10.6939 | 10 | 11 | 11 | 11 |
| MD | 335946 | 9.5985 | 9 | 10 | 10 | 10 |
| MA | 700745 | 20.0213 | 20 | 20 | 20 | 21 |
| NH | 214460 | 6.1274 | 6 | 6 | 6 | 7 |
| NJ | 241222 | 6.8921 | 6 | 7 | 7 | 7 |
| NY | 953043 | 27.2298 | 27 | 27 | 27 | 28 |
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$$\text{HM}(7,8) = 7.4666\cdots$$



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In a Round About Way

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$261818/7 = 37403$; over by 2403.

$261818/8 = 32727$, under by 2273.

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1840 Census

Apportionment based on the 1840 census used $d = 70680$ and **Webster's method** of rounding, yielding $h = 233$.

This was the only time in U.S. history that the House size decreased as a result of a census-based reapportionment.

The Vinton Act

The Vinton Act of 1850 (Representative Samuel Vinton, Whig-Ohio) was passed to head off politicizing the census figures. The idea was to adopt a permanent appropriation act.



The Vinton Act

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But experience exposed problems with the Vinton Act.

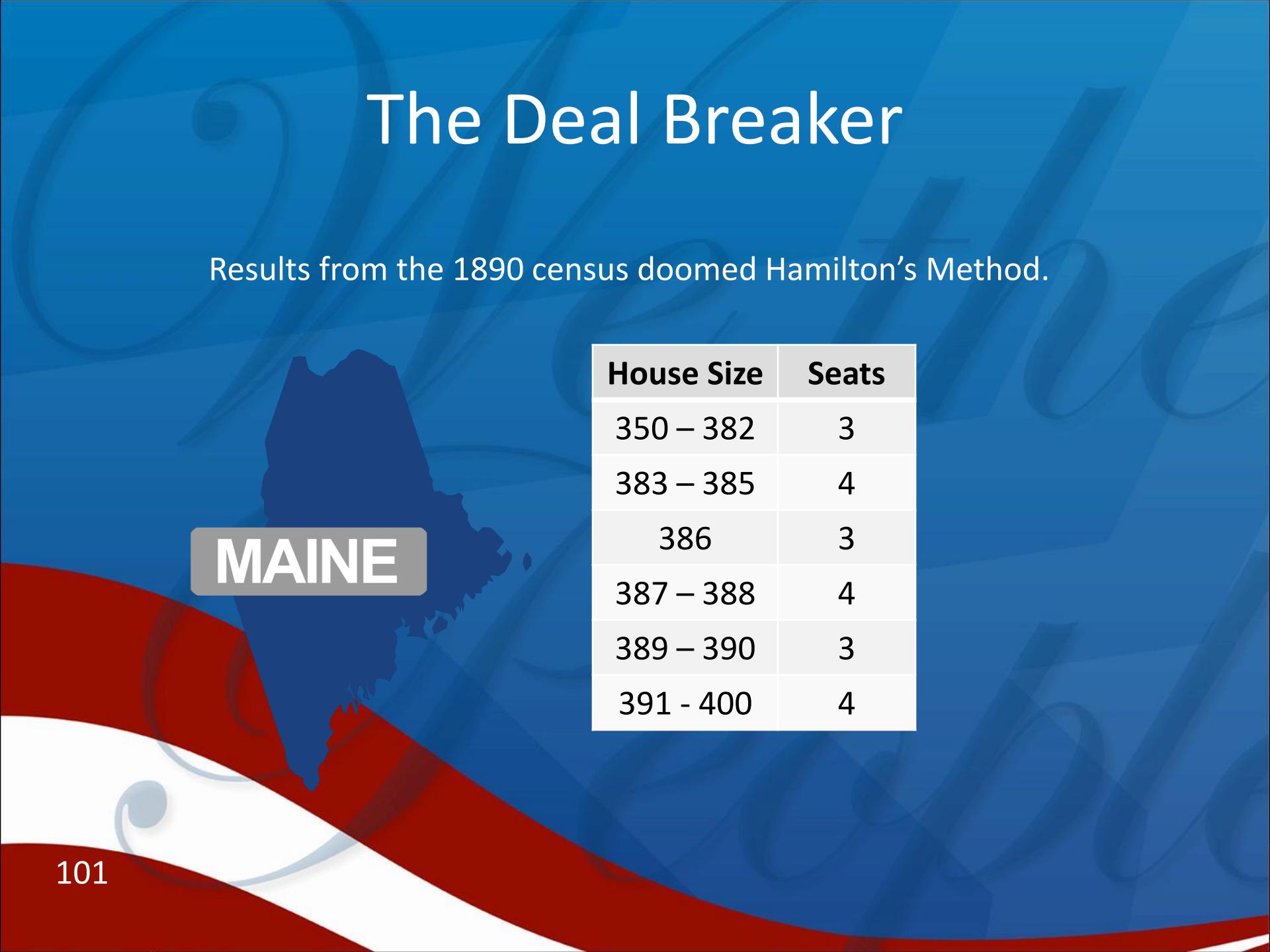
Alabama Paradox

This Paradox may occur with the Hamilton method:

when the number of House seats is increased, a given state's apportionment may decrease.

The Deal Breaker

Results from the 1890 census doomed Hamilton's Method.



MAINE

| House Size | Seats |
|------------|-------|
| 350 – 382 | 3 |
| 383 – 385 | 4 |
| 386 | 3 |
| 387 – 388 | 4 |
| 389 – 390 | 3 |
| 391 - 400 | 4 |

1910

Apportionment based on the 1910 census came from another mutation in apportionment methodology.

Congress abandoned the Quota Method and used a modified divisor method.

Modified Divisor Methods

Step 1. Select the House size, h .

Step 2. Apply a Basic Divisor Method to obtain h seats.

1910: $h = 433$ and Webster's method.

1920 Census

In the 1920 decade, for the only time in U. S. History, no census-based re-apportionment act was passed.

Congress could not agree on either the size of the House or on the method of apportionment. Further, the politics of prohibition played a significant role: the dries would not support any proposal that gave the wets more power.

Today

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Let $q = p/d$ and $n = \text{int}(q)$.

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Dean:

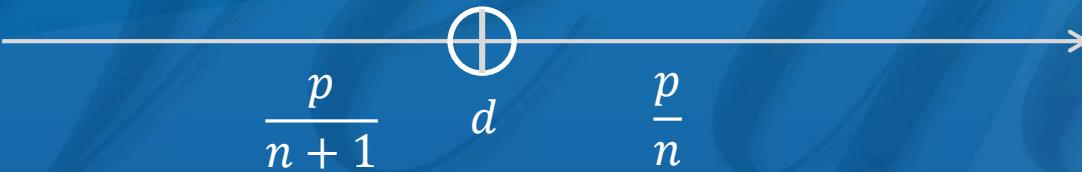
$$\frac{p}{n+1} \quad d \quad \frac{p}{n}$$


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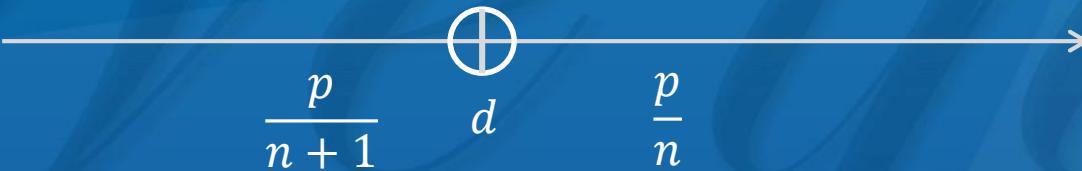


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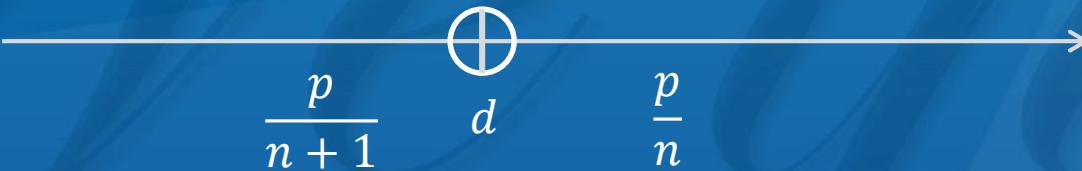
Criterion: $a = n+1$ iff $\frac{d}{\left(\frac{p}{n+1}\right)} \leq \frac{\left(\frac{p}{n}\right)}{d}$

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H-H:



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They Mean Well

A ***modified divisor method*** first fixes the House size, then seeks a divisor that when the state's quotients are rounded appropriately and summed, the house size is achieved.

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Webster: arithmetic mean $\frac{n + (n + 1)}{2} = n + \frac{1}{2}$

Huntington-Hill: geometric mean $\sqrt{n(n + 1)}$

Dean: harmonic mean $\frac{2}{\frac{1}{n} + \frac{1}{n + 1}} = \frac{2n(n + 1)}{2n + 1}$

Serial Distribution

- Step 1. Award 1 seat to each state.
Today this distributes 50 seats.
- Step 2. Then award the 51st seat, 52nd seat, 53rd seat, etc., according to a list of priority numbers.

Priority Numbers

$$A_n = \frac{\text{population}}{\text{ave}(n,n+1)}$$

Five Averages

- Greatest Divisors ave
- Harmonic Mean max
- Equal Proportions HM
- Major Fractions GM
- Smallest Divisors AM

The Last Seat

Who got the 435th seat?

| | ave | 435 |
|---------------------|-----|-----|
| • Greatest Divisors | max | IL |
| • Harmonic Mean | HM | MN |
| • Equal Proportions | GM | MN |
| • Major Fractions | AM | NC |
| • Smallest Divisors | min | WA |

The Last Seat

The 436th seat.

| | | 435 | 436 |
|---------------------|-----|-----|-----|
| • Greatest Divisors | max | IL | WA |
| • Harmonic Mean | HM | MN | CA |
| • Equal Proportions | GM | MN | NC |
| • Major Fractions | AM | NC | MO |
| • Smallest Divisors | min | WA | PA |

The Last Seat

The 436th seat.

| | | 435 | 436 |
|---------------------|-----|-----|-----|
| • Greatest Divisors | max | IL | WA |
| • Harmonic Mean | HM | MN | CA |
| • Equal Proportions | GM | MN | NC |
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The Aftermath

Michel Balinski, Professor of Mathematics at SUNY Stony Brook and H. Peyton Young, Professor of Mathematics at Johns Hopkins proved the following theorem in 1982:

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Michel Balinski, Professor of Mathematics at SUNY Stony Brook and H. Peyton Young, Professor of Mathematics at Johns Hopkins proved the following theorem in 1982:

There are no perfect apportionment methods.

Any method that satisfies the quota rule produces paradoxes; any method that is free of the Alabama paradox may violate the quota rule.

Distribution of Resources

Thank You

It is time that I took my seat in this House!

<http://www.nia977.wix.com/drbcap>

Priority Numbers

2010 Census

| Seat | Priority | State | Apportionment |
|------|----------|-------|---------------|
| 434 | 711308 | CA | 53 |
| 435 | 710231 | MN | 8 |
| 436 | 709063 | NC | 14 |
| 437 | 708459 | MO | 9 |
| 438 | 706337 | NY | 28 |
| 439 | 705164 | NJ | 13 |
| 440 | 703158 | MT | 2 |

[http://www.census.gov/data/tables/2010/dec
/2010-apportionment-data.html](http://www.census.gov/data/tables/2010/dec/2010-apportionment-data.html)