

# The Apportionment Problem

## Bringing Down the House

Charles Biles, Ph.D.  
Mathematics Colloquium  
Humboldt State University  
15 October 2015

website: [nia977.wix.com/drbcap](http://nia977.wix.com/drbcap)

“... no political problem is less susceptible of a precise solution than that which relates to the number most convenient for a representative legislature, ...”

James Madison  
*The Federalist 55*

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# Today's Program

## Four Parts:

1. Background for the CAP.
2. Basic Divisor Methods: 1790 – 1840.
3. Hamilton Quota Method: 1850 – 1900.
4. Modified Divisor Methods: 1910 – present.

# Part 1

## Background for the Congressional Apportionment Problem

# An Average Lecture

1. The average of  $a$  and  $b$  where  $0 < a < b$ .
2. How to round a positive decimal number.

# An Average Lecture

1. The average of  $a$  and  $b$  where  $0 < a < b$ .

$$\text{ave}(a,b) = \max(a,b) = b$$

$$\min(a,b) = a$$

$$\text{AM}(a,b) = (a + b)/2$$

$$\text{GM}(a,b) = \sqrt{a \times b}$$

$$\text{HM}(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a + b} = \frac{(\text{GM})^2}{\text{AM}}$$

# An Average Lecture

2. How to round a positive decimal number.

Let  $q > 0$  and  $n = \text{int}(q)$  where  $q - n > 0$ .

Let  $\text{rnd}(q) \in \{n, n+1\}$  where  $\text{rnd}(q) = n+1$  iff

$q > \max(n, n+1)$	round down
$q < \min(n, n+1)$	round up
$q = \text{AM}(n, n+1)$	round normally
$q = \text{GM}(n, n+1)$	geometric mean
$q = \text{HM}(n, n+1)$	harmonic mean

# The Congressional Apportionment Problem

Determine how many seats in the U.S.  
House of Representatives each state gets.

# CONGRESSIONAL SEATS

2010  
OFFICIAL RESULTS



(US apportionment population = 309,183,463)/435 ≈ 710,767

<http://www.census.gov/2010census/data/apportionment-data.php>

# The Constitution: Article I

**Section 1.** All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

# The Constitution: Article I

**Section 2.** The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, . . .

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Representatives . . . shall be apportioned among the several States . . . according to their respective Numbers, . . .

# The Constitution: Article I

**Section 2.** The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, . . .

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The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, . . .

# The Constitution: Article I

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The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, . . .

The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative; . . .

# The Apportionment Problem: Two Views

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- Transformation View:
- Distribution View:

# The Apportionment Problem: Two Views

- **Transformation View:** How to transform the census into seats in the House.
- **Distribution View:**

# The Apportionment Problem: Two Views

- **Transformation View:** How to transform the census into seats in the House.
- **Distribution View:** How to distribute the House seats among the States.

# The Apportionment Problem: Two Approaches

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- Constituency Approach:
- House Size Approach:

# The Apportionment Problem: Two Approaches

- **Constituency Approach:** How many people should a congressperson represent?
- **House Size Approach:**

# The Apportionment Problem: Two Approaches

- **Constituency Approach:** How many people should a congressperson represent?
- **House Size Approach:** How many seats should there be in the House?

# First Apportionment Bills

Census 1790

State	Population
CT	236841
DE	55540
GA	70835
KY	68705
MD	278514
MA	475327
NH	141822
NJ	179570
NY	331589
NC	353523
PA	432879
RI	68446
SC	206236
VT	85533
VA	630560
US	3615920

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3792621 — City of Los Angeles 2010

# First Apportionment Bills

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House Bill

30000

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DE	55540
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KY	68705
MD	278514
MA	475327
NH	141822
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NY	331589
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VT	85533
VA	630560
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House Bill

Divisor 30000

# First Apportionment Bills

Census 1790		House Bill
State	Population	Divisor 30000
CT	236841	7.895
DE	55540	1.851
GA	70835	2.361
KY	68705	2.290
MD	278514	9.284
MA	475327	15.844
NH	141822	4.727
NJ	179570	5.986
NY	331589	11.053
NC	353523	11.784
PA	432879	14.429
RI	68446	2.282
SC	206236	6.875
VT	85533	2.851
VA	630560	21.019
US	3615920	

# First Apportionment Bills

Census 1790

State	Population
CT	236841
DE	55540
GA	70835
KY	68705
MD	278514
MA	475327
NH	141822
NJ	179570
NY	331589
NC	353523
PA	432879
RI	68446
SC	206236
VT	85533
VA	630560
US	3615920

House Bill

Divisor 30000	Seats
7.895	7
1.851	1
2.361	2
2.290	2
9.284	9
15.844	15
4.727	4
5.986	5
11.053	11
11.784	11
14.429	14
2.282	2
6.875	6
2.851	2
21.019	21

# First Apportionment Bills

Census 1790		House Bill		
State	Population	Divisor	30000	Seats
CT	236841	7.895		7
DE	55540	1.851		1
GA	70835	2.361		2
KY	68705	2.290		2
MD	278514	9.284		9
MA	475327	15.844		15
NH	141822	4.727		4
NJ	179570	5.986		5
NY	331589	11.053		11
NC	353523	11.784		11
PA	432879	14.429		14
RI	68446	2.282		2
SC	206236	6.875		6
VT	85533	2.851		2
VA	630560	21.019		21
US	3615920	112		

# First Apportionment Bills

Census 1790		House Bill		Senate Bill	
State	Population	Divisor 30000	Seats	Divisor 33000	Seats
CT	236841	7.895	7	7.177	7
DE	55540	1.851	1	1.683	1
GA	70835	2.361	2	2.147	2
KY	68705	2.290	2	2.082	2
MD	278514	9.284	9	8.440	8
MA	475327	15.844	15	14.404	14
NH	141822	4.727	4	4.298	4
NJ	179570	5.986	5	5.442	5
NY	331589	11.053	11	10.048	10
NC	353523	11.784	11	10.713	10
PA	432879	14.429	14	13.118	13
RI	68446	2.282	2	2.074	2
SC	206236	6.875	6	6.250	6
VT	85533	2.851	2	2.592	2
VA	630560	21.019	21	19.108	19
US	3615920	112			

# First Apportionment Bills

Census 1790

State	Population
CT	236841
DE	55540
GA	70835
KY	68705
MD	278514
MA	475327
NH	141822
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RI	68446
SC	206236
VT	85533
VA	630560
US	3615920

House Bill

Divisor 30000	Seats
7.895	7
1.851	1
2.361	2
2.290	2
9.284	9
15.844	15
4.727	4
5.986	5
11.053	11
11.784	11
14.429	14
2.282	2
6.875	6
2.851	2
21.019	21
<b>112</b>	

Senate Bill

Divisor 33000	Seats
7.177	7
1.683	1
2.147	2
2.082	2
8.440	8
14.404	14
4.298	4
5.442	5
10.048	10
10.713	10
13.118	13
2.074	2
6.250	6
2.592	2
19.108	19
<b>105</b>	

# Rule of Three

Federalists in Congress apply a new idea:

Multiply the House size by each state's proportion to determine the state's **quota** (fair share of the House).

# Rule of Three

Federalists in Congress apply a new idea:

Multiply the House size by each state's proportion to determine the state's **quota** (fair share of the House).

$$\text{quota} = (\text{House size}) \times \frac{\text{state population}}{\text{national population}}$$

# The House Bill

Census

House Bill

State	Population	Divisor 30000	Seats
CT	236841	7.895	7
DE	55540	1.851	1
GA	70835	2.361	2
KY	68705	2.290	2
MD	278514	9.284	9
MA	475327	15.844	15
NH	141822	4.727	4
NJ	179570	5.986	5
NY	331589	11.053	11
NC	353523	11.784	11
PA	432879	14.429	14
RI	68446	2.282	2
SC	206236	6.875	6
VT	85533	2.851	2
VA	630560	21.019	21
US	3615920	112	

# The House Bill

Census

House Bill

<b>State</b>	<b>Population</b>	<b>Divisor 30000</b>	<b>Seats</b>	<b>Quota 112</b>
CT	236841	7.895	7	7.336
DE	55540	1.851	1	1.720
GA	70835	2.361	2	2.194
KY	68705	2.290	2	2.128
MD	278514	9.284	9	8.627
MA	475327	15.844	15	14.723
NH	141822	4.727	4	4.393
NJ	179570	5.986	5	5.562
NY	331589	11.053	11	10.271
NC	353523	11.784	11	10.950
PA	432879	14.429	14	13.408
RI	68446	2.282	2	2.120
SC	206236	6.875	6	6.388
VT	85533	2.851	2	2.649
VA	630560	21.019	21	19.531
US	3615920	112		112

# Problem

Census

House Bill

State	Population	Divisor 30000	Seats	Quota 112
CT	236841	7.895	7	7.336
DE	55540	1.851	1	1.720
GA	70835	2.361	2	2.194
KY	68705	2.290	2	2.128
MD	278514	9.284	9	8.627
MA	475327	15.844	15	14.723
NH	141822	4.727	4	4.393
NJ	179570	5.986	5	5.562
NY	331589	11.053	11	10.271
NC	353523	11.784	11	10.950
PA	432879	14.429	14	13.408
RI	68446	2.282	2	2.120
SC	206236	6.875	6	6.388
VT	85533	2.851	2	2.649
VA	630560	21.019	21	19.531
US	3615920	112		112

The Quota Rule  
is violated.

# The Senate Bill

Census

State	Population
CT	236841
DE	55540
GA	70835
KY	68705
MD	278514
MA	475327
NH	141822
NJ	179570
NY	331589
NC	353523
PA	432879
RI	68446
SC	206236
VT	85533
VA	630560
US	3615920

Senate Bill

Divisor 33000	Seats
7.177	7
1.683	1
2.147	2
2.082	2
8.440	8
14.404	14
4.298	4
5.442	5
10.048	10
10.713	10
13.118	13
2.074	2
6.250	6
2.592	2
19.108	19
105	

# The Senate Bill

Census

Senate Bill

State	Population	Divisor 33000	Seats	Quota 105
CT	236841	7.177	7	6.877
DE	55540	1.683	1	1.613
GA	70835	2.147	2	2.057
KY	68705	2.082	2	1.995
MD	278514	8.440	8	8.088
MA	475327	14.404	14	13.803
NH	141822	4.298	4	4.118
NJ	179570	5.442	5	5.214
NY	331589	10.048	10	9.629
NC	353523	10.713	10	10.266
PA	432879	13.118	13	12.570
RI	68446	2.074	2	1.988
SC	206236	6.250	6	5.989
VT	85533	2.592	2	2.484
VA	630560	19.108	19	18.310
US	3615920	105	105	105

# Problem

Census

Senate Bill

State	Population	Divisor 33000	Seats	Quota 105
CT	236841	7.177	7	6.877
DE	<b>55540</b>	<b>1.683</b>	<b>1</b>	<b>1.613</b>
GA	70835	2.147	2	2.057
KY	68705	2.082	2	1.995
MD	278514	8.440	8	8.088
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PA	432879	13.118	13	12.570
RI	68446	2.074	2	1.988
SC	206236	6.250	6	5.989
VT	85533	2.592	2	2.484
VA	<b>630560</b>	<b>19.108</b>	<b>19</b>	<b>18.310</b>
US	3615920	105	105	105

Large states are favored over small states.

# Hamilton's Method

State	Population
CT	236841
DE	55540
GA	70835
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VA	630560
US	3615920

120.5307

$d = 30000$

# Hamilton's Method

State	Population	
CT	236841	
DE	55540	
GA	70835	
KY	68705	
MD	278514	
MA	475327	
NH	141822	
NJ	179570	
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NC	353523	
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SC	206236	
VT	85533	
VA	630560	
US	3615920	

$h = 120$

$120.5307$

$d = 30000$

# Hamilton's Method

State	Population	$h = 120$	Quota
CT	236841		7.860
DE	55540		1.843
GA	70835		2.351
KY	68705		2.280
MD	278514		9.243
MA	475327		15.774
NH	141822		4.707
NJ	179570		5.959
NY	331589		11.004
NC	353523		11.732
PA	432879		14.366
RI	68446		2.271
SC	206236		6.844
VT	85533		2.839
VA	630560		20.926
US	3615920	120.5307	120

$d = 30000$

# Hamilton's Method

State	Population	$h = 120$	Quota	Lower Q
CT	236841		7.860	7
DE	55540		1.843	1
GA	70835		2.351	2
KY	68705		2.280	2
MD	278514		9.243	9
MA	475327		15.774	15
NH	141822		4.707	4
NJ	179570		5.959	5
NY	331589		11.004	11
NC	353523		11.732	11
PA	432879		14.366	14
RI	68446		2.271	2
SC	206236		6.844	6
VT	85533		2.839	2
VA	630560		20.926	20
US	3615920	<b>120.5307</b>	120	111

$d = 30000$

# Hamilton's Method

State	Population	$h = 120$	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
NJ	179570		5.959	5	6
NY	331589		11.004	11	11
NC	353523		11.732	11	12
PA	432879		14.366	14	14
RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

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CT	236841		7.860	7	8
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MA	475327		15.774	15	16
NH	141822		4.707	4	5
NJ	179570		5.959	5	6
NY	331589		11.004	11	11
NC	353523		11.732	11	12
PA	432879		14.366	14	14
RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

This became the first apportionment bill passed by Congress.

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PA	432879		14.366	14	14
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26 March 1792:  
bill is sent to President Washington for his approval.

# Hamilton's Method

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MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
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NY	331589		11.004	11	11
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PA	432879		14.366	14	14
RI	68446		2.271	2	2
SC	206236		6.844	6	7
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VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

This became the first apportionment bill passed by Congress.

26 March 1792:  
bill is sent to President Washington for his approval.

5 April 1792: Washington vetoes the bill.

# Hamilton's Method

State	Population	$h = 120$	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
MA	475327		15.774	15	16
NH	141822		4.707	4	5
NJ	179570		5.959	5	6
NY	331589		11.004	11	11
NC	353523		11.732	11	12
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SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

U.S.:

$$3615920/120 = 30,132.66\dots$$

# Hamilton's Method

State	Population	$h = 120$	Quota	Lower Q	Appt
CT	236841		7.860	7	8
DE	55540		1.843	1	2
GA	70835		2.351	2	2
KY	68705		2.280	2	2
MD	278514		9.243	9	9
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NH	141822		4.707	4	5
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RI	68446		2.271	2	2
SC	206236		6.844	6	7
VT	85533		2.839	2	3
VA	630560		20.926	20	21
US	3615920	120.5307	120	111	120

Connecticut:  
 $236841/8 = 29605.13$ .

Delaware:  
 $55540/2 = 27770$

U.S.:  
 $3615920/120 = 30,132.66...$

# Basic Jefferson Method

After Washington's veto on 5 April 1792, Congress quickly passed the original Senate bill. Washington signed the bill on 14 April 1792.

# Two Methodologies

- Divisor Methods
- Quota Methods

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Divisor methods **create** seats.

Quota methods **distribute** seats.

# Part 2

## Basic Divisor Methods

1790 - 1840

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Note that  $h = \sum_1^s a_i$ .

# Basic Divisor Methods

Step 1: Select  $d \in \mathbb{N}_{30000}$ .

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Result:  $h = \sum_1^s a_i$ .

# First 60 years

- A Basic Divisor Method would be used as the House apportionment method until 1850.
  - ❖ 1790:  $s = 15$ ;  $d = 33000 \Rightarrow h = 105$
  - ❖ 1800:  $s = 16$ ;  $d = 33000 \Rightarrow h = 141$
  - ❖ 1810:  $s = 17$ ;  $d = 35000 \Rightarrow h = 181$
  - ❖ 1820:  $s = 24$ ;  $d = 40000 \Rightarrow h = 213$
  - ❖ 1830:  $s = 24$ ;  $d = 47700 \Rightarrow h = 240$
  - ❖ 1840:  $s = 26$ ;  $d = 70680 \Rightarrow h = 223$

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Webster: round normally.

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A constituency of 46,776 is closer to the target constituency of 50,000; hence, Dean awards Vermont 6 seats.

# James Dean (1830 census)

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Hence,  $a_i = n_i + 1$  iff  $d - \frac{p_i}{n_i+1} < \frac{p_i}{n_i} - d$ .

# Dean's Criterion

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# Daniel Webster

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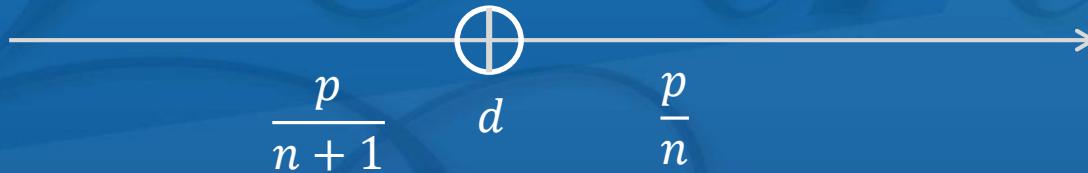
# Dean and Webster

Step 1: Select the constituency,  $d$ .

Step 2: Calculate  $q = p/d$  and  $n = \text{int}(q)$ .

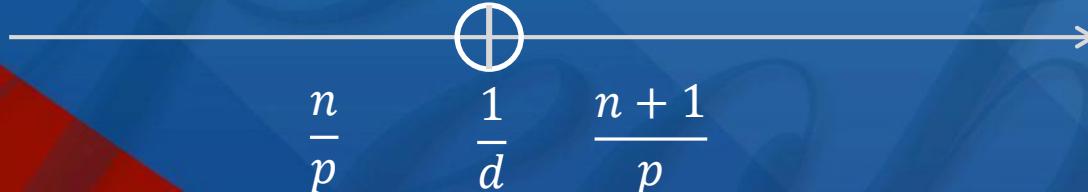
Step 3: Let the apportionment be either  $n$  or  $n+1$ ,  
with  $n+1$  iff

Dean:



**Dean:**  $a = n+1 \Leftrightarrow \text{HM}(n, n+1) < q.$

Webster:



**Webster:**  $a = n+1 \Leftrightarrow \text{AM}(n, n+1) < q.$

# In a Round About Way

Census 1810		$d = 35000$				
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DE	71004	2.0287	2	2	2	3
GA	210346	6.0099	6	6	6	7
KY	374287	10.6939	10	11	11	11
MD	335946	9.5985	9	10	10	10
MA	700745	20.0213	20	20	20	21
NH	214460	6.1274	6	6	6	7
NJ	241222	6.8921	6	7	7	7
NY	953043	27.2298	27	27	27	28
NC	487971	13.9420	13	14	14	14
OH	230760	6.5931	6	7	7	7
PA	809773	23.1364	23	23	23	24
RI	76931	2.1980	2	2	2	3
SC	336569	9.6163	9	10	10	10
TN	243913	6.9689	6	7	7	7
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VA	817594	23.3598	23	23	23	24
US	6575234	188.1222	181	188	189	198

# In a Round About Way

$$\text{HM}(7,8) = 7.4666\cdots$$



Census 1810		$d = 35000$				
State	Population	Quotient	Jefferson	Webster	Dean	Adams
CT	261818	7.4805	7	7	8	8
DE	71004	2.0287	2	2	2	3
GA	210346	6.0099	6	6	6	7
KY	374287	10.6939	10	11	11	11
MD	335946	9.5985	9	10	10	10
MA	700745	20.0213	20	20	20	21
NH	214460	6.1274	6	6	6	7
NJ	241222	6.8921	6	7	7	7
NY	953043	27.2298	27	27	27	28
NC	487971	13.9420	13	14	14	14
OH	230760	6.5931	6	7	7	7
PA	809773	23.1364	23	23	23	24
RI	76931	2.1980	2	2	2	3
SC	336569	9.6163	9	10	10	10
TN	243913	6.9689	6	7	7	7
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# In a Round About Way

Given  $q = p/d$  and  $n = \text{int}(q)$  where  $q - n > 0$ .  
Note:  $n < q < n+1$ .

Then  $a \in \{n, n + 1\}$  where  $a = n + 1$  iff

Jefferson:  $q > \max(n, n+1)$

Dean:  $q > \text{HM}(n, n+1)$

Webster:  $q > \text{AM}(n, n+1)$

Adams:  $q > \min(n, n+1)$

# 1840 Census

The Apportionment Act of 1842 used a basic divisor method with  $d = 70680$  and **Webster's method** of rounding, yielding  $h = 233$ .

This was the only time in U.S. history that the House size decreased as a result of a census-based re-apportionment.

# Part 2

## Quota Methods

1850 - 1900

# Quota Method

Step 1. Let  $h \in \mathbb{N}_s$ .

Step 2. Calculate  $Q_i = h(p_i/p)$ ,  $n_i = \text{int}(Q_i)$ .

Step 3. Select  $a_i \in \{n_i, n_i+1\}$  so that  $h = \sum_1^s a_i$ .

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Example: California, 2010 Census

$$Q_{CA} = h(p_{CA}/p_{US}) = 435 \left( \frac{37,341,989}{309,183,463} \right) = 52.5376\dots$$

<http://www.census.gov/prod/cen2010/briefs/c2010br-08.pdf>

# The Vinton Act

The Vinton Act of 1850 (Representative Samuel Vinton, Whig-Ohio) was passed to head off politicizing the census and to adopt a permanent appropriation act.



# The Vinton Act

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But experience exposed problems with the Vinton Act.

# Lessons from History

The quota method displayed counter-intuitive paradoxes; especially, the Alabama Paradox:

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when the number of House seats is increased,  
a state's apportion may decrease.

# Alabama Paradox

Results from the 1900 census doomed Hamilton's method—Maine oscillated as follows:

3 members for  $h = 350-382, 386, 389-390$

4 members for  $h = 383-385, 387-388, 391-400$

# Part 3

Modified Divisor Methods

1910 - present

# Modified Divisor Methods

- Step 1. Select  $h \in \mathbb{N}_s$ .
- Step 2. Select  $d \in \mathbb{N}_{30000}$ .
- Step 3. Calculate  $q_i = p_i/d$  and  $n_i = \text{int}(q_i)$ .
- Step 4. Let  $a_i = \text{rnd}(q_i) \in \{n_i, n_i+1\}$
- Step 5. If  $\sum_1^s a_i = h$ , then DONE;  
else, modify  $d$  and GO TO Step 3.

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1830: Jefferson:  $a_i = n_i$

Adams:  $a_i = n_i + 1$

Dean:  $a_i = n_i + 1$  iff  $q_i > \text{HM}(n_i, n_i+1)$

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Step 5. If  $\sum_1^s a_i = h$ , then DONE;  
else, modify  $d$  and GO TO Step 3.

1910 :  $h = 433$  and Webster's method of rounding.

# 1920 Census

In the 1920 decade, for the only time in U. S. history, no census-based re-apportionment act was passed.

Congress could not agree on either House size or method of apportionment. And, the politics of prohibition played a huge role: the dries would not support any proposal that gave the wets more power.

# Today

The current method consists of the Apportionment Act of 1929 (which froze  $h = 435$ ) along with its 1940 and 1941 amendments. The 1941 amendment was signed by President Franklin Roosevelt and specifies the apportionment method of

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# Huntington-Hill

Let  $q_i = p_i/d$  and  $n_i = \text{int}(q_i)$ .

Then  $a_i = n_i + 1$  iff  $q_i >$

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Dean:

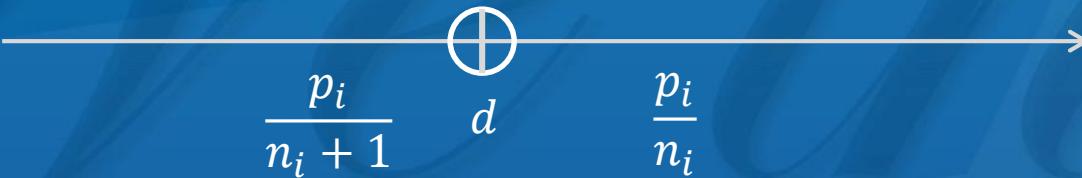
$$\frac{p_i}{n_i + 1} \oplus d \oplus \frac{p_i}{n_i}$$

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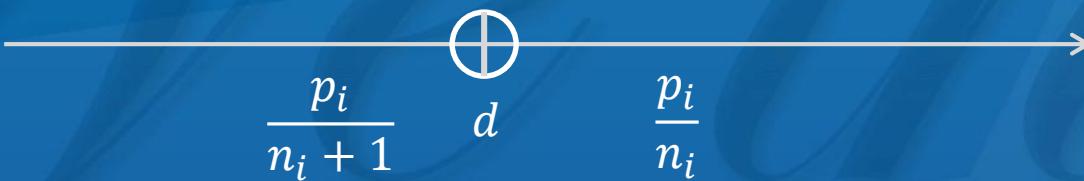


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Dean:



H-H:



Criterion:  $a_i = n_i + 1$  iff  $\frac{d}{\left(\frac{p_i}{n_i + 1}\right)} < \frac{\left(\frac{p_i}{n_i}\right)}{d}$

# H-H Criterion

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$$\begin{aligned} a_i = n_i + 1 &\Leftrightarrow \frac{d}{\left(\frac{p_i}{n_i+1}\right)} < \frac{\left(\frac{p_i}{n_i}\right)}{d} \\ &\Leftrightarrow d^2 < \frac{p_i^2}{n_i(n_i+1)} \\ &\Leftrightarrow n_i(n_i + 1) < \left(\frac{p_i}{d}\right)^2 \end{aligned}$$

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# They Mean Well

A ***modified divisor method*** first fixes the House size, then seeks a divisor that when the state's quotients are rounded appropriately and summed, the house size is achieved.

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Webster: arithmetic mean  $\frac{n + (n + 1)}{2} = n + \frac{1}{2}$

Huntington-Hill: geometric mean  $\sqrt{n(n + 1)}$

Dean: harmonic mean  $\frac{2}{\frac{1}{n} + \frac{1}{n + 1}} = \frac{2n(n + 1)}{2n + 1}$

# The Aftermath

Michel Balinski, Professor of Mathematics at SUNY Stony Brook and H. Peyton Young, Professor of Mathematics at Johns Hopkins proved the following theorem in 1982:

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Michel Balinski, Professor of Mathematics at SUNY Stony Brook and H. Peyton Young, Professor of Mathematics at Johns Hopkins proved the following theorem in 1982:

*There are no perfect apportionment methods.*

In particular, any divisor method may violate the quota rule, and any quota method produces paradoxes.

# The Apportionment Question

How many seats in the U.S.  
House of Representatives  
does each state get?

# Priority Numbers

$$PN_i(n) = p_i / \text{ave}(n, n+1)$$

where  $\text{ave}(n, n+1) =$

Jefferson:  $\max(n, n+1)$

Dean:  $\text{HM}(n, n+1)$

Huntington-Hill:  $\text{GM}(n, n+1)$

Webster:  $\text{AM}(n, n+1)$

Adams:  $\min(n, n+1)$

# Thank You

It is time that I took my seat in this House!

For more: <http://nia977.wix.com/drbcap>

# Priority Numbers

Census 1790		Seats	Priority Numbers H-H			
State	Population		sqrt(2)	sqrt(6)	sqrt(12)	sqrt(20)
Connecticut	236841	1	167471	96689	68370	52959
Delaware	55540	1	39272	22674	16033	12419
Georgia	70835	1	50087	28918	20448	15839
Kentucky	68705	1	48581	28048	19833	15362
Maryland	278514	1	196939	113702	80400	62277
Massachusetts	475327	1	336106	194051	137215	106286
New Hampshire	141822	1	100283	57898	40940	31712
New Jersey	179570	1	126975	73309	51837	40153
New York	331589	1	234468	135370	95721	74145
North Carolina	353523	1	249978	144325	102053	79050
Pennsylvania	432879	1	306091	176722	124961	96794
Rhode Island	68446	1	48398	27942	19758	15304
South Carolina	206236	1	145830	84195	59535	46115
Vermont	85533	1	60480	34918	24691	19125
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	North Carolina	353523	1		144325	102053	79050
	Pennsylvania	432879	2		176722	124961	96794
	Rhode Island	68446	1	48398	27942	19758	15304
	South Carolina	206236	1	145830	84195	59535	46115
	Vermont	85533	1	60480	34918	24691	19125
	Virginia	630560	3			<b>182026</b>	140997
	US	3615920	23				

# Reform

Three Proposals:

- Thirty-thousand.org
- The Wyoming Rule
- Neubauer and Gartner

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CA: 1244 seats!

# The Wyoming Rule

The Wyoming Rule is a basic divisor method in which the divisor is the population of the least populous state (currently WY; hence, the name).

[http://en.wikipedia.org/wiki/Wyoming\\_Rule](http://en.wikipedia.org/wiki/Wyoming_Rule)

<http://www.outsidethebeltway.com/representation-in-the-house-the-wyoming-rule/>

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Let's apply the WY Rule to the 2000  
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2010: smallest state: WY, 563626

$h = 543$  Dean HI

$h = 542$  Huntington-Hill

$h = 540$  Webster NJ, SD

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A Proposal for Apportioning the House

Michael G. Neubauer, CSU Northridge, Mathematics

Margo G. Gartner (master's degree student)

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